ON THE WAVE RESISTANCE AND LIFT OF A PLANE CONTOUR OF ARBITRARY FORM IN UNSTEADY MOTION UNDER A FREE SURFACE

(O VOLNOVOM SOPROTIVLENII I POD'EMNOI SILE PLOSKOGO CONTURA PROIZVOL'NOI FORMY PRI NEUSTANOVIVSHEMSIA DVIZHENII POD SVOBODNOI POVERKHNOST'IU)

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The problem of calculating the wave resistance of a cylinder moving under a free surface in unsteady motion has been previously solved by Sretenskii [1] and Havelock [2]. Expressions for the complex velocity and the forces for the unsteady motion of an arbitrary plane contour under a free surface are calculated below.

1. Let a plane contour of arbitrary form be set in motion from a state of rest with the velocity $v(t)$ under the free surface of a heavy ideal incompressible fluid.

We shall choose a coordinate system such that the origin of the coordinates is located in the horizontal plane which is coincident with the initial undisturbed level. The x-axis is in the direction of the motion and the y-axis is directed vertically upwards. We shall consider the fluid motion to be potential and the waves which arise on the free surface to be small. The latter assumption is equivalent to the contour being deeply submerged in the fluid.

To find the absolute potential $\varphi(x, y, t)$ of the disturbed velocities of the moving contour in a moving coordinate system we have the Laplace equation $\Delta\varphi = 0$ with the boundary conditions:

at the free surface for $y = 0$

$$\frac{\partial^2\varphi}{\partial t^2} + g\frac{\partial\varphi}{\partial y} + v^2(t)\frac{\partial^2\varphi}{\partial x^2} - 2v(t)\frac{\partial^2\varphi}{\partial x\,\partial t} - \frac{dv(t)}{dt}\frac{\partial\varphi}{\partial x} = 0 \tag{1.1}$$

at the surface of the body

$$\frac{\partial\varphi}{\partial n} = v_n(t) \tag{1.2}$$

at infinity

$$\varphi(x, y, t) \to 0 \quad \text{for } x \to +\infty, \qquad \varphi(x, y, t) \to 0 \quad \text{for } y \to -\infty \tag{1.3}$$

As initial conditions one can take the condition that the motion of the contour commences without initial velocity and, in addition, that the fluid is found at rest with its surface horizontal. Indeed, for the problem under consideration the equation of the free surface will be determined by the formula

$$\zeta(x, t) = \frac{1}{g}\left(\frac{\partial\varphi}{\partial t} + v(t)\frac{\partial\varphi}{\partial x}\right) \qquad \text{for } y = 0 \tag{1.4}$$

At the initial moment of time $t = 0$ let the contour be found in a state of rest, i.e. let $v(0) = 0$; then for $t = 0$

$$\zeta(x, 0) = \frac{1}{g}\frac{\partial\varphi}{\partial t}\bigg|_{y=0}$$

Thus, in order that the fluid at the initial moment of time $t = 0$ be found at relative rest and that its free surface be horizontal, it is necessary that

$$\frac{\partial\varphi}{\partial t}\bigg|_{y=0} = 0, \ \varphi\,|_{y=0} = 0 \qquad \text{for } t = 0 \tag{1.5}$$

As a preliminary we shall consider the particular cases of the unsteady motion of singularities of the type of plane vortices, sources, etc.

2. Let a plane vortex with constant circulation Γ at depth h under the free surface be moved from a state of rest. The velocity potential $\varphi(x, y, t)$ of the absolute motion is then found from the solution of the equation $\Delta\varphi = 0$ with condition (1.1) and the boundary conditions at infinity. We shall form the expression

$$\varphi = \varphi_1 - \varphi_2 + \varphi_3 \qquad \left(\varphi_1 = -\frac{\Gamma}{2\pi}\tan^{-1}\frac{y+h}{x}, \ \varphi_2 = \frac{\Gamma}{2\pi}\tan^{-1}\frac{y-h}{x}\right) \tag{2.1}$$

in which φ_3 is for the present an unknown function. It is obvious that for $y = 0$

$$\frac{\partial}{\partial x}(\varphi_1 - \varphi_2) = 0, \qquad \frac{\partial^2}{\partial x^2}(\varphi_1 - \varphi_2) = 0, \qquad \frac{\partial}{\partial y}(\varphi_1 - \varphi_2) = -2\frac{\partial\varphi_2}{\partial y}$$

Then equation (1.1) for $y = 0$ taking (2.1) into consideration takes the form

$$\frac{\partial^2\varphi_3}{\partial x^2} - 2v(t)\frac{\partial\varphi_3}{\partial x\partial t} + v^2(t)\frac{\partial^2\varphi_3}{\partial x^2} + g\frac{\partial\varphi_3}{\partial y} - \frac{dv(t)}{\partial t}\frac{\partial\varphi_3}{\partial x} = 2g\frac{\partial\varphi_2}{\partial y} \tag{2.2}$$

In all that follows we shall write v instead of $v(t)$. Since

$$\frac{\partial\varphi_2}{\partial y} = \frac{\Gamma}{2\pi}\frac{x}{x^2+(y-h)^2}$$

the equality

$$\frac{\partial\varphi_2}{\partial y} = -\frac{\Gamma}{2\pi}\int_0^\infty e^{k(y-h)}\sin kx dx \tag{2.3}$$

will then be valid for $y = 0$.

Then equation (2.2) for $y = 0$ will have the form

$$\frac{\partial^2\varphi_3}{\partial t^2} - 2v\frac{\partial^2\varphi_3}{\partial x\partial t} + v^2\frac{\partial^2\varphi_3}{\partial x^2} + g\frac{\partial\varphi_3}{\partial y} - \frac{dv}{dt}\frac{\partial\varphi_3}{\partial x} = -2g\frac{\Gamma}{2\pi}\int_0^\infty e^{k(y-h)}\sin kx\, dk \tag{2.4}$$

We shall consider the equation

$$\frac{\partial^2 u}{\partial t^2} - 2v\frac{\partial^2 u}{\partial t\partial x} + v^2\frac{\partial^2 u}{\partial x^2} + g\frac{\partial u}{\partial y} - \frac{dv}{dt}\frac{\partial u}{\partial x} = -\frac{g\Gamma}{\pi}\int_0^\infty e^{k[(y-h)+ix]}dk \tag{2.5}$$

It is obvious that, if u is determined, φ_3 is then found from the condition $\varphi_3 = \operatorname{Im} u$.

We shall seek a solution of equation (2.5) in the form

$$u = \frac{1}{\pi}\int_0^\infty A(k,t)\, e^{k[(y-h)+ix]}dk \tag{2.6}$$

Substituting (2.6) into (2.5) and comparing the left and right sides, we find the condition for determining $A(k, t)$

$$\frac{d^2A}{dt^2} - 2ikv\frac{dv}{dt} - \left[v^2k^2 - gk + ik\frac{dv}{dt}\right]A = -g\Gamma \tag{2.7}$$

We shall introduce the new variable

$$B = A\exp\left(-ik\int_0^t v d\tau\right)$$

Substituting the latter equality into (2.7), we obtain

$$B'' + gkB = -g\Gamma\exp\left(-ik\int_0^t v d\tau\right) \tag{2.8}$$

The characteristic equation of (2.8) is

$$m^2 + gk = 0, \qquad m_{1,2} = \pm i\sqrt{gk}$$

We shall take the particular solution of the inhomogeneous linear equation in the form

$$B = -g\Gamma \int_0^t \exp\left(-ik \int_0^\tau v\,d\tau\right) \frac{\sin[\sqrt{gk}(t-\tau)]}{\sqrt{gk}}\,d\tau \tag{2.9}$$

We then obtain

$$A = -g\Gamma \int_0^t \exp\left(ik \int_\tau^t v\,d\tau\right) \frac{\sin[\sqrt{gk}(t-\tau)]}{\sqrt{gk}}\,d\tau \tag{2.10}$$

Substituting (2.10) into (2.6) we obtain

$$u = -\frac{g\Gamma}{\pi} \int_0^\infty \int_0^t \exp\left\{k\left[(y-h) + i\left(x + \int_\tau^t v\,d\tau\right)\right]\right\} \frac{\sin[\sqrt{gk}(t-\tau)]}{\sqrt{gk}}\,d\tau\,dk \tag{2.11}$$

Hence

$$\varphi_3 = \operatorname{Im} u = -\frac{g\Gamma}{\pi} \int_0^\infty \int_0^t \frac{1}{\sqrt{gk}} e^{k(y-h)} \sin k\left[x + \int_\tau^t v\,d\tau\right] \sin\sqrt{gk}(t-\tau)\,d\tau\,dk \tag{2.12}$$

We shall now determine the form of the stream function through the equations

$$\partial\varphi / \partial x = \partial\psi / \partial y, \qquad \partial\varphi / \partial y = -\partial\psi / \partial x$$

For calculating ψ_3 we have the equality

$$\psi_3 = \int \frac{\partial\varphi_3}{\partial y}\,dx + Q(y) \tag{2.13}$$

where $Q(y)$ is for the present an unknown function of y. It can be found from the condition

$$\frac{\partial\varphi_3}{\partial x} = -\int \frac{\partial^2\varphi_3}{\partial y^2}\,dx + \theta'(y) \tag{2.14}$$

After carrying out the calculation, we obtain that $Q'(y) = 0$. We shall set $\theta(y)$ = const = 0. Consequently,

$$\psi_3 = -\frac{g\Gamma}{\pi} \int_0^\infty \int_0^t \frac{1}{\sqrt{gk}} e^{k(y-h)} \cos\left[k\left(x + \int_\tau^t v\,d\tau\right)\right] \sin[\sqrt{gk}(t-\tau)]\,d\tau dk \tag{2.15}$$

We shall construct an expression for the complex potential $\omega_3(z, t) = \varphi_3 + i\psi_3$ where $z = x + iy$ of a vortex in unsteady motion

$$\omega_3(z,t) = \frac{g\Gamma}{\pi i} \int_0^\infty \int_0^t \frac{e^{-ik[z-ih]}}{\sqrt{gk}} \exp\left(-ik \int_\tau^t v\,d\tau\right) \sin[\sqrt{gk}(t-\tau)]\,d\tau\,dk \tag{2.16}$$

The total complex potential for the motion of the vortex is

$$W(z,t)=\frac{\Gamma}{2\pi i}\ln\frac{z+ih}{z-ih}+\frac{g\Gamma}{\pi i}\int_0^\infty\int_0^t\frac{e^{-ik(z-ih)}}{\sqrt{gk}}\exp\left(-ik\int_\tau^t v\,d\tau\right)\sin\left[\sqrt{gk}\,(t-\tau)\right]d\tau\,dk \tag{2.17}$$

From (2.17) it is seen that the expression obtained satisfies both the boundary conditions and the initial conditions. Indeed, assuming that the vortex is found in a state of rest at the initial moment of time ($v(0) = 0$), we obtain from (2.17)

$$\zeta(x,0)=\frac{1}{g}\frac{\partial\varphi}{\partial t}=0,\qquad \varphi=0\qquad \text{for } t=0,\ y=0$$

i.e. at the initial moment of time the free surface of the fluid is horizontal and the motion of the fluid commences without initial velocity.

3. Let a plane source of strength $Q(t)$, which is being moved from a state of rest in an unsteady manner along the x-axis, be found at the point $(0, -h)$.

We shall assume that at the initial moment of time the strength of the source is equal to $Q(0) = 0$. The velocity potential $\varphi(x, y, t)$ of the absolute motion is found from the solution of the equation $\Delta\varphi = 0$ with condition (1.1) and the boundary conditions at infinity. As initial conditions one can take the condition that for $t = 0$ the free surface is initially found at rest in its horizontal position of equilibrium.

To find the disturbed velocity potential $\varphi(x, y, t)$ we construct the expression

$$\varphi=\varphi_1-\varphi_2+\varphi_3\qquad\left(\varphi_1=\frac{Q(t)}{2\pi}\ln\sqrt{x^2+(y+h)^2},\ \varphi_2=\frac{Q(t)}{2\pi}\ln\sqrt{x^2+(y-h)^2}\right) \tag{3.1}$$

Taking (3.1) into account equation (1.1) will have the form

$$\frac{\partial^2\varphi_3}{\partial t^2}-2v\frac{\partial^2\varphi_3}{\partial x\,\partial t}+v^2\frac{\partial^2\varphi_3}{\partial x^2}+g\frac{\partial\varphi_3}{\partial y}-\frac{dv}{dt}\frac{\partial\varphi_3}{\partial x}=\frac{gQ(t)}{\pi}\int_l^\infty e^{k(y-h)}\cos kx\,dk \tag{3.2}$$

Solving (3.2) in a manner analogous to that set forth in Section 2, we obtain

$$\varphi_3=\frac{g}{\pi}\int_0^\infty\int_0^t\frac{Q(\tau)}{\sqrt{gk}}e^{k(y-h)}\cos\left[k\left(x+\int_\tau^t v\,d\tau\right)\right]\sin\left[\sqrt{gk}\,(t-\tau)\right]dk\,d\tau \tag{3.3}$$

It is easy to verify that (3.1) satisfies the initial conditions of the problem. Indeed, taking (3.3) into account it is seen from formula (3.1) that if $Q(0) = 0$ for $t = 0$ the motion of the fluid then arises from a state of rest. If $Q(0) = Q_0 \neq 0$ for $t = 0$, we shall then have at the initial moment

$$\varphi(x, y, 0) = \frac{Q_0}{2\pi}\left(\ln\sqrt{x^2 + (y+h)^2} - \ln\sqrt{x^2 + (y-h)^2}\right) \tag{3.4}$$

The equation of the free surface at the initial moment will be determined by (1.4). Substituting (3.4) into (1.4), we obtain that $\zeta(x, 0) = 0$, i.e. in this case at the initial moment of time $t = 0$ the free surface is found at rest in its horizontal position. We shall find an expression for the stream function from condition (2.13).

Carrying out the calculation, we obtain

$$\psi_3 = \frac{g}{\pi}\int_0^\infty\int_0^t \frac{Q(\tau)}{\sqrt{gk}} e^{k(y-h)} \sin\left[k\left(x + \int_\tau^t v\,d\tau\right)\right] \sin\left[\sqrt{gk}(t-\tau)\right] dk\,d\tau \tag{3.5}$$

$\omega_3(z, t)$ will then have the form

$$\omega_3(z, t) = \frac{g}{\pi}\int_0^\infty\int_0^t \frac{Q(\tau)}{\sqrt{gk}} e^{-ik[z-ih]} \exp\left(-ik\int_0^t v\,d\tau\right) \sin\left[\sqrt{gk}(t-\tau)\right] dk\,d\tau \tag{3.6}$$

The complex potential for the unsteady motion of a source will have the form

$$W(z, t) = \frac{Q(t)}{2\pi}\ln\left[(z+ih)(z-ih)\right] +$$
$$+\frac{g}{\pi}\int_0^\infty\int_0^t \frac{Q(t)}{\sqrt{gk}} e^{-ik[z-ih]} \exp\left(-ik\int_\tau^t v\,d\tau\right) \sin\left[\sqrt{gk}(g-\tau)\right] dk\,d\tau \tag{3.7}$$

Differentiating (3.7) with respect to z, we obtain an expression for the complex potential of a doublet with axis parallel to the x-axis

$$W(z, t) = \frac{Q(t)}{2\pi}\left[\frac{1}{z+ih} - \frac{1}{z-ih}\right] +$$
$$+\frac{1}{\pi i}\int_0^\infty\int_0^t Q(t)\sqrt{gk}\, e^{-ik(z-ih)} \exp\left(-ik\int_\tau^t v\,d\tau\right) \sin\sqrt{gk}(t-\tau)\, dk\,d\tau \tag{3.8}$$

By virtue of the linearity of the boundary conditions the expression for the complex potential of a vortex-source in the case of an unbounded fluid has the form

$$W(z) = \frac{\Gamma}{2\pi i}\ln(z+ih) + \frac{Q(t)}{2\pi}\ln(z+ih) = \frac{\Gamma + Q(t)\,i}{2\pi i}\ln(z+ih) \tag{3.9}$$

The complex potential of the unsteady motion of a vortex-source under a free surface will then have the form

$$W(z, t) = \frac{B}{2\pi i}\ln(z - z_1) - \frac{\overline{B}}{2\pi i}\ln(z - \overline{z}_1) +$$

$$+\frac{g}{\pi i}\int_0^\infty\int_0^t \frac{e^{-ik(z-\bar{z}_1)}}{\sqrt{gk}}\bar{B}\exp\left(-ik\int_\tau^t v d\tau\right)\sin\sqrt{gk}\,(t-\tau)\,dkd\tau \tag{3.10}$$

Here the notation

$$\Gamma + iQ = B, \qquad \Gamma - iQ = B, \qquad x + ih = z - z_1$$

has been introduced.

4. We shall calculate the complex potential $W(z, t) = \varphi + i\psi$ for the unsteady motion of an arbitrary contour with velocity $v(t)$ under the free surface.

We shall assume that the motion of the contour commences from a state of rest. In this case the problem of finding $W(z, t)$ is reduced to finding analytic functions $\varphi(x, y, t)$ and $\psi(x, y, t)$ in the lower half-plane according to the equation $\Delta\varphi = 0$ with the conditions (1.1), (1.2) and (1.3). As initial conditions we again take the condition that at $t = 0$ the fluid is found in a state of relative rest in the horizontal position, i.e.

$$(dW / dt)_{y=0} = 0, \qquad \varphi = 0 \quad \text{at } t = 0$$

We shall designate by $W_0(z, t)$ the complex velocity potential of the flow which is obtained for the unsteady motion of the contour C in an unbounded fluid.

We take an arbitrary contour C_1 within which the body is found. Then, for an arbitrary point z lying between the contour and the body the value $W_0(z, t)$ according to the Cauchy formula will be

$$W_0(z, t) = \frac{1}{2\pi i}\int_{C_1}\frac{W_0(\zeta, t)}{\zeta - z}\,d\zeta \tag{4.1}$$

where the variable of integration, which runs around the whole contour of integration, is denoted by ζ. According to the Cauchy formula the expression for the complex velocity will be

$$\frac{dW_0}{dz} = \frac{1}{2\pi i}\int_{C_1}\frac{dW_0}{d\zeta}\frac{d\zeta}{\zeta - z} = \frac{1}{2\pi i}\int_{C^1}\frac{dW_0}{\zeta - z} \tag{4.2}$$

Let $dW_0(\zeta, t) = d\varphi_0 + id\psi_0$. From formula (3.9) it is seen that the complex velocity of a vortex-source will be

$$\bar{v}(z, t) = \frac{\Gamma + iQ}{2\pi i}\frac{1}{z - \zeta}$$

If the last expression $\bar{v}(z, t)$ be compared with formula (4.2), one can imagine that the fluid motion takes place as a result of the presence of a series of vortex sources on the contour C. Moreover, a vortex $d\varphi_0$ and a source $d\psi_0$ are located on each elementary length $d\zeta$ of the contour. Then, replacing $B = \Gamma + iQ$ by $dW_0(z, t)$ in formula (3.10) and carrying out the integration along the contour C, we obtain an expression for the complex velocity from the vortex-sources distributed along the contour.

This wave motion can be regarded in the first approximation as that which would have been generated by the motion of the contour C. The approximate aspect of the solution is a result of the fact that the strength of the vortices and sources for the motion of the contour under a free surface are different from the values which Γ and Q would have for the motion of the contour in an unbounded flow. This change will be the smaller the deeper the contour is immersed in the fluid.

In the first approximation the solution of the problem of the unsteady motion of the contour C under the free surface can be determined by the formula

$$\frac{dW(z,t)}{dz} = \left(\frac{dW(z,t)}{dt}\right)_1 + \left(\frac{dW(z,t)}{dt}\right)_2 \tag{4.3}$$

Here

$$\left(\frac{dW}{dz}\right)_1 = \frac{1}{2\pi i}\int\limits_C \frac{dW_0(\zeta,t)}{z-\zeta} \tag{4.4}$$

$$\left(\frac{dW}{dz}\right)_2 = \frac{1}{2\pi i}\int\limits_C \frac{d\overline{W}_0(\zeta,t)}{z-\zeta} -$$

$$-\frac{1}{\pi}\int\limits_C d\overline{W}_0(\zeta,t)\int\limits_0^\infty \sqrt{gk}e^{-ik(z-\bar{\zeta})}\,dk\int\limits_0^t \exp\left(-ik\int\limits_\tau^t v\,d\tau\right)\sin\sqrt{gk}\,(t-\tau)\,d\tau \tag{4.5}$$

The solutions (4.3), (4.4) and (4.5) satisfy all the conditions of the problem, including the initial conditions also, since, for $t = 0$, W_0 is equal to zero.

The forces which act on the body can be calculated from the formula

$$X + iY = i\rho\int\limits_C \frac{dW}{dt}\,dz_1 + \frac{\rho i}{2}\frac{dv(t)}{dt}\int\limits_C (z - z_1)\,dz_1 + \frac{i\rho}{2}\int\limits_C \left(\frac{dW}{dz}\right)^2 dz\Big] \tag{4.6}$$

where X and Y denote the projections of the force on the x- and y-axis, and $z_1 = x - iy$.

Formula (4.6) was obtained by Sretenskii [1] and follows as a special case of the more general formula of Chaplygin [3]. For calculating the forces, as has been shown by Kochin [4], the complex velocity for an unbounded fluid $\bar{v}(z) = (dW_0/dz)_\infty$ can be taken in the first approximation in place of $\bar{v}(z) = dW_0/dz$. This class of forces is not considered in this paper.

The wave resistance of an arbitrary contour and its lift of wave nature will be determined by the third term of (4.6), i.e.

$$X + iY = \frac{i\rho}{2}\oint_C \left(\frac{dW}{dz}\right)_0^2 dz \tag{4.7}$$

In formula (4.7) the derivative $(dW_0/dz)_0$ is the complex velocity in the relative motion; therefore

$$\left(\frac{dW}{dz}\right)_0 = \left(\frac{dW}{dz}\right)_1 + \left(\frac{dW}{dz}\right)_2 - v(t)$$

We shall compute the integral

$$J = \oint_{C_1} \left(\frac{dW}{dz}\right)_0^2 dz =$$

$$= \oint_{C_1} \left(\frac{dW}{dz}\right)_1^2 dz + \oint_{C_1} \left[\left(\frac{dW}{dz}\right)_2 - v(t)\right]^2 dz + 2 \oint_{C_1} \left(\frac{dW}{dz}\right)_1 \left[\left(\frac{dW}{dz}\right)_2 - v(t)\right] dz \tag{4.8}$$

The first two integrals here vanish, i.e.

$$\oint_{C_1} \left(\frac{dW}{dz}\right)_1^2 dz = 0, \quad \oint_{C_1} \left[\left(\frac{dW}{dz}\right)_2 - v(t)\right]^2 dz = 0$$

since the function $(dW/dz)_1$ is holomorphic on and exterior to the contour C_1 and has a zero of at least first order at infinity, and the function $(dW(z,\ t)/dz)_2$ is holomorphic on and interior to the contour C_1. The equality (4.8) can be represented yet again as

$$J = 2\oint_{C_1} \left[\left(\frac{dW}{dz}\right)_1 + \left(\frac{dW}{dz}\right)_2\right]\left[\left(\frac{dW}{dz}\right)_2 - v(t)\right] dz = 2\oint_{C_1} \frac{dW}{dz}\left[\left(\frac{dW}{dz}\right)_2 - v(t)\right] dz \tag{4.9}$$

$$\oint_{C_1} \left(\frac{dW}{dz}\right)_2 \left[\left(\frac{dW}{dz}\right)_2 - v(t)\right] dz = 0$$

Consequently,

$$J = 2\oint_{C_1} \frac{dW}{dz}\left(\frac{dW}{dz}\right)_2 dz - 2v(t)\oint_{C_1} \frac{dW}{dz}\, dz = 2\oint_{C_1} \frac{dW}{dz}\left(\frac{dW}{dz}\right)_2 dz - 2v(t)\,\Gamma$$

$$\left(\Gamma = \oint_{C_1} \frac{dW}{dz}\, dz\right) \tag{4.10}$$

Here Γ is the circulation of the velocity about the contour C. Making use of expression (4.5), we obtain

$$\int\limits_{C_1} \frac{dW}{dz}\left(\frac{dW}{dz}\right)_2 dz = \frac{1}{2\pi i}\int\limits_{C_1}\int\limits_{C} \frac{dW(z,t)}{dz}\frac{\overline{dW(\zeta,t)}}{d\bar\zeta}\left[\frac{1}{z-\bar\zeta} - \right.$$

$$\left. -2i\int\limits_0^\infty \sqrt{gk}e^{-ik(z-\bar\zeta)}\,dk\int\limits_0^t \exp\left(-ik\int\limits_\tau^t v\,d\tau\right)\sin\sqrt{gk}(t-\tau)\,d\tau\right]dz\,d\bar\zeta \tag{4.11}$$

Substituting (4.11) into (4.7), we obtain

$$X+iY = \frac{\rho}{2\pi}\int\limits_{C}\int\limits_{C_1} \frac{dW}{dz}\frac{d\overline{W}}{d\bar\zeta}\left[\frac{1}{z-\bar\zeta} - 2i\int\limits_0^\infty \sqrt{gk}e^{-ik(z-\bar\zeta)}\,dk\,\times\right.$$

$$\left.\times\int\limits_0^t \exp\left(ik\int\limits_\tau^t v\,d\tau\right)\sin\sqrt{gk}(t-\tau)\,d\tau\right]dz\,d\bar\zeta - i\rho v(t)\,\Gamma \tag{4.12}$$

We shall transform the right-hand side of (4.12), If the points ζ and z belong to the lower half-plane, the following equality is valid:

$$\frac{1}{z-\bar\zeta} = i\int\limits_0^\infty e^{-ik(z-\bar\zeta)}\,dk$$

Then the first integral term of (4.12) takes the form

$$\int\limits_{C_1}\int\limits_{C}\frac{dW}{dz}\frac{d\overline{W}}{d\bar\zeta}\frac{1}{z-\bar\zeta}\,dz\,d\bar\zeta = \int\limits_{C_1}\int\limits_{C}\int\limits_0^\infty \frac{dW}{dz}\frac{d\overline{W}}{d\bar\zeta}e^{-ikz}e^{ik\bar\zeta}\,dk\,dz\,d\bar\zeta =$$

$$= i\int\limits_0^\infty H(k,t)\overline{H(k,t)}\,dk = i\int\limits_0^\infty |H(k,t)|^2\,dk \qquad \left(H(k,t) = \int\limits_C e^{-ikz}\frac{dW}{dz}\,dz\right) \tag{4.13}$$

An arbitrary contour of the lower half-plane which encircles the contour C in the positive direction can be taken as the contour of integration.

The remaining part of (4.12) is written in the following form:

$$-2i\int\limits_{C_1}\int\limits_{C}\frac{dW}{dz}\frac{d\overline{W}}{d\bar\zeta}\int\limits_0^\infty \sqrt{gk}e^{-ik(z-\bar\zeta)}\,dk\int\limits_0^t \exp\left(-ik\int\limits_\tau^t v\,d\tau\right)\sin\sqrt{gk}(t-\tau)\,d\tau\,dz\,d\bar\zeta =$$

$$= -2i\int\limits_0^\infty |H(k,t)|^2\sqrt{gk}\,dk\int\limits_0^t \exp\left(-ik\int\limits_\tau^t v\,d\tau\right)\sin\sqrt{gk}(t-\tau)\,d\tau \tag{4.14}$$

Consequently

$$X+iY = \frac{i\rho}{2\pi}\int\limits_0^\infty |H(k,t)|^2\,dk - \frac{i\rho}{\pi}\int\limits_0^\infty |H(k,t)|^2\sqrt{gk}\,dk\,\times$$

$$\times \int_0^t \exp\left(-ik\int_\tau^t v\,d\tau\right)\sin\sqrt{gk}\,(t-\tau)\,d\tau - i\rho v(t)\,\Gamma \tag{4.15}$$

The wave resistance is then

$$R = \frac{\rho}{\pi}\int_0^\infty |H(k,t)|^2\sqrt{gk}\,dk\int_0^t \sin\left(k\int_\tau^t v\,d\tau\right)\sin\sqrt{gk}\,(t-\tau)\,d\tau \tag{4.16}$$

$$Y = \rho g S + \rho v(t)\,\Gamma - \frac{\rho}{2\pi}\int_0^\infty |H(k,t)|^2\,dk + \tag{4.17}$$

$$+\frac{\rho}{\pi}\int_0^\infty |H(k,t)|^2\sqrt{gk}\,dk\int_0^t \cos\left(k\int_\tau^t v\,d\tau\right)\cos\sqrt{gk}\,(t-\tau)\,d\tau$$

Here $\rho g S$ is Archimedes' displacement force; S is the area which envelops the contour.

In conclusion we shall find the wave resistance and the lift of a cylinder of radius r_0 with circulation Γ about the contour of the cylinder in unsteady motion at a depth h under the free surface. The complex velocity corresponding to the motion in an unbounded fluid has the form

$$\left(\frac{dW}{dz}\right)_\infty = \frac{v(\tau)\,r_0^2}{(z+ih)^2} + \frac{\Gamma}{2\pi i\,(z+ih)} \tag{4.18}$$

Then

$$H(k) = \int_C e^{-ikz}\left(\frac{dW}{dz}\right)_\infty dz = e^{-\nu h}\left[\Gamma + 2\pi v(\tau)\,k r_0^2\right] \tag{4.19}$$

Substituting (4.19) into (4.16), we obtain

$$R = \frac{\rho}{\pi}\int_0^\infty e^{-2kh}\sqrt{gk}\,[\Gamma + 2\pi v(\tau)\,k r_0^2]^2\,dk\int_0^t \sin\left(k\int_\tau^t v\,d\tau\right)\sin\sqrt{gk}\,(t-\tau)\,d\tau \tag{4.20}$$

For $\Gamma = 0$ we obtain the wave resistance of a cylinder

$$R = 4\pi\rho g r_0^4\int_0^\infty e^{-2kh}k^2\sqrt{gk}\,dk\int_0^t v^2(\tau)\sin\left(k\int_\tau^t v\,d\tau\right)\sin\sqrt{gk}\,(t-\tau)\,d\tau \tag{4.21}$$

The latter formula is analogous to the formula of Sretenskii [1].

Indeed, replacing $v(\tau)$ by $\sqrt{(gk)}/k$ and, in addition, replacing the sines in the inner integral by cosines with the same arguments (this leads only to a change in sign), we obtain an expression for the wave resistance of a cylinder in the form of Sretenskii.

Setting the radius r_0 equal to zero in (4.20), we obtain the wave resistance of a vortex

$$R = \frac{\rho\Gamma^2}{\pi}\cdot\int_0^\infty e^{-2kh}\sqrt{gk}\,dk\int_0^t \sin\left(k\int_\tau^t v\,d\tau\right)\sin\sqrt{gk}\,(t-\tau)\,d\tau \qquad (4.22)$$

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BIBLIOGRAPHY

1. Sretenskii, L.N., K teorii volnovogo soprotivleniia (On the theory of wave resistance). *Tr. TsAGI*, No. 458, 1939.

2. Havelock, T.H., The wave resistance of a cylindrical body, moved from statical. *Quart. J. Mech. and Appl. Math.*, Vol. 2.

3. Chaplygin, S.A., *Sobranie sochinenii (Collected Works)*, Vol. 2. Gostekhizdat, 1948.

4. Kochin, N.E., *O volnovom soprotivlenii i pod''emnoi sile pogruzhenykh v zhidkost' tel (On the wave resistance and lift of bodies immersed in a fluid)*. Sobranie sochinenii (Collected works), Vol. 2, Akad. Nauk SSSR, 1949.

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